A Unified Approach to Nonparametric Spectrum Estimation Algorithms

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Abstract—**Different approaches to spectrum estimation can be broadly classified as parametric and nonparametric methods. In the parametric techniques, an underlying model is assumed in the formulation of the spectrum estimation problem and one estimates the parameters of the model. For nonparametric methods, no such model is assumed. In this paper, several nonparametric spectrum estimation algorithms are brought under a unified framework by the introduction of a generalized nonparametric spectrum estimation algorithm. A four-stage approach is employed. It contains as special cases the Blackman-Tukey algorithm, the weighted, overlapped segment averaging (WOSA) method, the lag-reshape approach, Rader's algorithm, and the short-time unbiased spectrum estimation (STUSE) algorithm. The framework proposed in the paper is more general than the one recently proposed by Nuttall and Carter. Theoretical expressions for the spectrum estimation variance of the generalized algorithm are derived, and then verified via simulation example. Also, several nonparametric approaches for obtaining unbiased spectrum estimates are discussed and compared. Finally we conclude the paper with a brief discussion of the applicability and usefulness of several methods in specific situations.**

I. Introduction

In the last two decades, much research effort in solving the spectrum estimation problem for discrete time series has been directed toward parametric techniques [7], [8], [12], [16], [17], [34]. Such approaches are based on modeling the data by a small set of parameters $\{a_1, a_2, \cdots, a_q\}$ and finding appropriate values for the q parameters by some numerical scheme. The attractiveness of parametric spectrum estimation algorithms has been mainly due to their superior spectral resolution. However, in many situations, the process of characterizing the spectrum by parameters is not physically justified and attempts to represent the spectrum by a few parameters can produce misleading results. As a result, nonparametric spectrum estimation methods have continued to receive attention [18], [21], [28], [29], [33].

In this paper we present a generalized nonparametric spectrum estimation algorithm, which provides a unified framework for nonparametric spectrum estimation methods. As the name suggests, nonparametric methods of spectrum estimation differ from the parametric ones in that no specific model is assumed in the formulation of the problem. We will show how several nonparametric methods available in literature fit into this unified framework. The discussion will emphasize theoretical aspects and deal with spectrum estimation bias and stability, spectral resolution, and spectral leakage suppression capability of the generalized method.

We are concerned with both autopower and cross-spectral density estimation, and therefore, we consider two discrete-time random processes $\{x_1(n)\}$ and $\{x_2(n)\}$ which are at least jointly wide-sense stationary. As is almost always the case in practice, we will deal with estimation of cross-spectral density from single time-limited realizations of each of these processes, denoted by $\{x_1(n)\}$ and $\{x_2(n)\}$, respectively.

Before introducing the generalized nonparametric spectrum estimation alogrithm, we will very briefly discuss (Section II) several nonparametric methods that we seek to bring under one unified framework. These include the Blackman-Tukey (BT) approach [5], the weighted, overlapped, segment averaging (WOSA) algorithm [11], [24], [25], [35], and Rader's technique [31], along with the more recent lag-reshape (LR) [10], [27], [28] and short-time unbiased spectrum estimation (STUSE) [2], [21], [29], [30], [36] algorithms. The recent method introduced by Thomson [33], where the power spectrum of a time series is estimated in terms of the solution to a basic integral equation that defines the Fourier transform of the time series and also Konvalinka's iterative nonparametric spectrum estimation algorithm [18], is beyond the scope of this framework.

In most problems in digital signal processing involving spectrum and correlation function estimates, it is important that the estimates are unbiased. The extreme importance of unbiased spectrum estimates has been demonstrated in [36] for applications of time delay estimation and magnitude-squared coherence (MSC) function estimation. Carter [9] has derived an analytical expression for the bias introduced in MSC function estimation when one of the signals involved is a delayed (and possibly noisy) version of the other. In this paper we also discuss several methods of obtaining unbiased nonparametric spectrum estimates. For this discussion, we will assume that the cross-correlation function of the signals is zero outside

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the range of lags that are used for the spectrum estimation. Therefore, by "unbiased," we imply that the cross-correlation function estimates are unbiased at those lags inside the range of interest. We will also compare several unbiased spectrum estimation algorithms in terms of estimation stability and spectral leakage suppression capability.

II. Brief Overview

Let $\{x_1(n)\}$ and $\{x_2(n)\}$ denote time-limited realizations of the discrete time processes $\{\chi_1(n)\}$ and $\{\chi_2(n)\}$, respectively. They are assumed to be at least jointly wide-sense stationary and to be of length P samples (i.e., $n = 0, 1, \cdots, P-1$).

In the classical Blackman–Tukey approach, a K-point estimate of the cross-spectral density of $\{\chi_1(n)\}$ and $\{\chi_2(n)\}$ was made as follows; we will first estimate $2L + 1$ values of the cross-correlation function $c_{12}(m)$ as

$$\hat{c}_{12}(m) = \frac{1}{P}\sum_{n=0}^{P-1} x_1(n)\, x_2(n-m); \qquad -L \leqslant n \leqslant L \tag{1}$$

and obtain the cross-spectral estimate as the K-point discrete Fourier transform (DFT) of the above estimates multiplied by a lag window function $w_3(m)$. That is,

$$\hat{G}_{12}^{(b)}(k\Delta_K) = \frac{1}{w_3(0)}\sum_{m=-L}^{L} \hat{c}_{12}(m)\, w_3(m)\, e^{-j(2\pi/K)mk}; \quad 0 \leqslant k \leqslant K-1, \quad K \geqslant 2L+1. \tag{2}$$

In the above expressions and in what follows, carets (^) denote estimated quantities. Also, we emphasize the fact that spectrum estimates are commonly made at uniform sampled frequencies in the range $[0, 1)$ by using the discrete frequency index Δ_K, where $k\Delta_K$ denotes (k/K) Hz. Extension of the results in the paper to the case when spectrum estimates are made at nonuniform samples of frequency is straightforward.

In the WOSA algorithm, we will partition $\{x_1(n)\}$ and $\{x_2(n)\}$ into N possibly overlapped segments of length L_1 each, apply a linear window function $w_1(n)$ multiplicatively to each segment, and compute the K-point DFT of the lth weighted segment of each sequence to obtain

$$X_{i,l}(k\Delta_K) = \sum_{n=0}^{L_1-1} x_i(lR+n)\, w_1(n)\, e^{-j(2\pi/K)nk}; \quad 0 \leqslant l \leqslant N-1, \quad 0 \leqslant k \leqslant K-1, \quad K \geqslant 2L_1, \quad i = 1 \text{ or } 2, \tag{3}$$

where R is the shift between adjacent segments in number of samples ($L_1 - R$ is the overlap between segments). Now, the K-point cross-spectral density estimate using the WOSA algorithm is obtained from (3) as the average of the sample cross-spectral density of the lth weighted segments of $\{x_1(n)\}$ and $\{x_2(n)\}$, and is given by

$$\hat{G}_{12}^{(w)}(k\Delta_K) = \frac{1}{r_{11}(0)}\frac{1}{K}\frac{1}{N}\sum_{l=0}^{N-1} X_{1,l}(k\Delta_K)\, X_{2,l}^*(k\Delta_K) \tag{4}$$

where superscript * denotes complex conjugate and $r_{11}(m)$ is the autocorrelation function of the window function defined as

$$r_{11}(m) = \frac{1}{K}\sum_{n=0}^{L_1-1} w_1(n)\, w_1(n-m). \tag{5}$$

The advantage of the WOSA method is due to its computational simplicity, since one can use fast Fourier transform (FFT) algorithms [1], [13], [14] to compute the DFT's of different segments.

For the STUSE approach we proceed by partitioning, windowing, and discrete Fourier transforming as for the WOSA method. However, we will now compute a sample cross-spectral density of the lth weighted segment of $\{x_1(n)\}$ and the $(l+q)$th weighted segment of $\{x_2(n)\}$ and average them over all possible values of l as

$$\hat{G}_{12,q}(k\Delta_K) = \frac{1}{N}\frac{1}{K}\sum_{l=0}^{N-1} X_{1,l}(k\Delta_K)\, X_{2,l+q}^*(k\Delta_K) \tag{6}$$

for different values of q ranging from Q_1 to Q_2.

From (6) we can obtain, for each q, an estimate of the true cross-spectral density by multiplying $\hat{G}_{12,q}(k\Delta_K)$ by $e^{j(2\pi/K)kqR}$, which would compensate for the shift between the lth and $(l+q)$th segments. The short-time unbiased spectrum estimate is now obtained as the weighted average of all such estimates and is given by

$$\hat{G}_{12}^{(s)}(k\Delta_K) = \frac{1}{\bar{r}_{11}(0)}\sum_{q=Q_1}^{Q_2} \hat{G}_{12,q}(k\Delta_K)\, e^{j(2\pi/K)kqR} \tag{7}$$

where

$$\bar{r}_{11}(m) = \sum_{q=Q_1}^{Q_2} r_{11}(m+qR). \tag{8}$$

Probably the easiest way to explain the lag-reshape algorithm is to say that the LR spectrum estimate is obtained by applying a quadratic window function $W_3(k\Delta_K)$ to the K-point WOSA spectrum estimate obtained in (4) using a linear window function $w_1(n)$. That is,

$$\hat{G}_{12}^{(l)}(k\Delta_K) = \frac{1}{w_3(0)}\hat{G}_{12}^{(w)}(k\Delta_K) \circledast W_3(k\Delta_K) \tag{9}$$

where $w_3(m)$ and $W_3(k\Delta_K)$ are discrete Fourier transform pairs and $\circledast$ denotes complex convolution. Note that when the parameters are properly interpreted, both the WOSA and Blackman–Tukey algorithms are special cases of the lag-reshape method. This generalization was introduced by Nuttall and Carter [27], [28].

The following is a general version of the Rader's algorithm for cross-spectral density estimation.

Partition $\{x_1(n)\}$ and $\{x_2(n)\}$ into N nonoverlapped segments of length L_1 each, and compute the $2L_1$ point DFT of each of these segments (no linear windowing is employed). Let $X_{1,l}(k\Delta_{2L_1})$ and $X_{2,l}(k\Delta_{2L_1})$ denote these DFT's. Now, compute two intermediate estimates of the cross-spectral density as ($K = 2L_1$ here)

$$\hat{G}'_{12}(k\Delta_K) = \frac{1}{L_1}\sum_{l=0}^{N-1}\left[\left\{X_{1,l}(k\Delta_K) + (-1)^k X_{1,l+1}(k\Delta_K)\right\} X^*_{2,l}(k\Delta_K)\right] \quad (10)$$

and

$$\hat{G}''_{12}(\Delta_K) = \frac{1}{L_1}\sum_{l=0}^{N-1}\left[\left\{X_{1,l}(k\Delta_K) + (-1)^k X_{1,l-1}(k\Delta_K)\right\} X^*_{2,l}(k\Delta_K)\right] \quad (11)$$

where $X_{1,l}(k\Delta_K)$ is defined to be zero for $l = -1$ and $l = N$.

It is straightforward to show that the correlation function estimate obtained as

$$\tilde{c}_{12}(m) = \begin{cases} \tilde{c}'_{12}(m); & -L_1 \leqslant m < 0 \\ \tilde{c}''_{12}(m); & 0 \leqslant m \leqslant L_1 - 1 \end{cases} \quad (12)$$

where $\tilde{c}'_{12}(m)$ and $\tilde{c}''_{12}(m)$ are the inverse DFT's of $\tilde{G}'_{12}(k\Delta_K)$ and $\tilde{G}''_{12}(k\Delta_K)$, respectively, is an unbiased estimate of the cross-correlation function of $\{x_1(n)\}$ and $\{x_2(n)\}$. Rader's spectrum estimate is given by applying a lag window function $w_3(m)$ to $\tilde{c}_{12}(m)$ and computing its DFT as

$$\hat{G}^{(r)}_{12}(k\Delta_K) = \sum_{m=-L_1}^{L_1-1} \tilde{c}_{12}(m)\, w_3(m)\, e^{-j(2\pi/K)km}. \quad (13)$$

The rest of the paper is organized as follows. In Section III, we will introduce the generalized nonparametric spectrum estimation algorithm and show how the five nonparametric spectrum estimation algorithms discussed above fit into this framework. In Section IV, we will discuss the spectrum estimation bias and variance for the generalized algorithm. We will also discuss several methods of obtaining unbiased spectrum estimates in this section. This section also contains some simulation results verifying the theoretical results. The spectral resolution and spectral leakage suppression properties of nonparametric spectrum estimation algorithms are discussed in Section V. Finally we make the concluding remarks in Section VI.

III. The Generalized Nonparametric Spectrum Estimation Algorithm

Let $\{x_1(n)\}$ and $\{x_2(n)\}$ be realizations of two jointly wide-sense stationary discrete-time processes $\{\chi_1(n)\}$ and $\{\chi_2(n)\}$, respectively, time-limited to P data samples. The generalized nonparametric spectrum estimation algorithm for computing the M-point cross-PDS estimate of $\{\chi_1(n)\}$ and $\{\chi_2(n)\}$ consists of the following four stages.

Stage 1: First, partition $\{x_1(n)\}$ and $\{x_2(n)\}$ into N possibly overlapped segments of length K each. Let $w_1(n)$ and $w_2(n)$ be two linear window functions which may or may not be identical. Apply $w_1(n)$ multiplicatively to each segment of $x_1(n)$ to obtain

$$x_{1,l}(n) = x_1(lR + n)\, w_1(n); \quad 0 \leqslant n \leqslant K-1, \quad 0 \leqslant l \leqslant N-1 \quad (14)$$

where R is the shift between adjacent segments. Similarly, weight each segment of $x_2(n)$ and $w_2(n)$ to obtain

$$x_{2,l}(n) = x_2(lR + n)\, w_2(n); \quad 0 \leqslant n \leqslant K-1, \quad 0 \leqslant l \leqslant N-1. \quad (15)$$

Padding zeros to the segments may be taken care of by defining the window functions appropriately.

Now compute the K-point sample cross-spectral density of the lth segment of $x_1(n)$ and the $(l+q)$th segment of $x_2(n)$ as

$$\hat{G}_{12,l,q}(k\Delta_K) = \frac{1}{K} X_{1,l}(k\Delta_K)\, X^*_{2,l+q}(k\Delta_K); \quad 0 \leqslant k \leqslant K-1, \quad q = Q_1, Q_1+1, \cdots, Q_2 \quad (16)$$

where $X_{i,j}(k\Delta_K)$ is the K-point DFT of $\{x_{i,j}(n)\}$ for $i = 1$, or 2 and $j = 0, 1, \cdots, N-1$ and $X_{2,l}(k\Delta_K)$ is defined to be zero for values of l that are smaller than zero and larger than $N-1$. The first stage spectrum estimate is now computed as the average of $\hat{G}_{12,l,q}(k\Delta_K)$ over all segments. That is,

$$\hat{G}_{12,q}(k\Delta_K) = \frac{1}{N}\sum_{l=0}^{N-1} \hat{G}_{12,l,q}(k\Delta_K). \quad (17)$$

Stage 2: An estimate of the cross-spectral density of $\{x_1(n)\}$ and $\{x_2(n)\}$ can be obtained from each of $\hat{G}_{12,q}(k\Delta_K)$ by weighting it with a complex exponential that will compensate for the shift between the lth segment of $x_1(n)$ and $(l+q)$th segment of $x_2(n)$. The second stage spectrum estimate is obtained by adding these weighted estimates of the spectrum over all q's as

$$\tilde{G}_{12}(k\Delta_K) = \sum_{q=Q_1}^{Q_2} \hat{G}_{12,q}(k\Delta_K)\, e^{j(2\pi/K)kqR}. \quad (18)$$

Stage 3: The third stage spectrum estimate is obtained as a frequency smoothed version of the second stage estimate. That is,

$$\hat{G}_{12}(k\Delta_K) = \tilde{G}_{12}(k\Delta_K) \circledast W_3(k\Delta_K) \quad (19)$$

$$= \frac{1}{K}\sum_{\mu=0}^{K-1} \tilde{G}_{12}(\mu\Delta_K)\, W_3((k-\mu)\Delta_K), \quad (20)$$

where $W_3(k\Delta_K)$ is a quadratic window function and for $k - \mu < 0$, $W_3((k-\mu)\Delta_K) = W_3((k-\mu+K)\Delta_K)$. Let $w_3(m)$ be the lag window function corresponding to the quadratic window function $W_3(k\Delta_K)$ obtained as the K-point inverse DFT of $W_3(k\Delta_K)$, i.e.,

$$w_3(m) = \frac{1}{K}\sum_{k=0}^{K-1} W_3(k\Delta_K)\, e^{j(2\pi/K)km}; \qquad \frac{-K}{2} \leqslant m < \frac{K}{2}. \tag{21}$$

$\hat{G}_{12}(k\Delta_K)$ may equivalently be computed in the following manner.

Let $\bar{c}_{12}(m)$ be the cross-correlation function estimate for $\{x_1(n)\}$ and $\{x_2(n)\}$ obtained as the K-point inverse DFT of $\bar{G}_{12}(k\Delta_K)$. That is,

$$\bar{c}_{12}(m) = \frac{1}{K}\sum_{k=0}^{K-1} \bar{G}_{12}(k\Delta_K)\, e^{j(2\pi/K)km}; \qquad \frac{-K}{2} \leqslant m < \frac{K}{2}. \tag{22}$$

Then the third stage spectrum estimate $\bar{G}_{12}(k\Delta_K)$ can be obtained as the K-point DFT of $\bar{c}_{12}(m)$ windowed with $w_3(m)$, i.e.,

$$\bar{G}_{12}(k\Delta_K) = \sum_{m=-K/2}^{K/2-1} \hat{c}_{12}(m)\, e^{-j(2\pi/K)km} \tag{23}$$

where

$$\hat{c}_{12}(m) = \bar{c}_{12}(m)\, w_3(m). \tag{24}$$

Stage 4: The fourth stage cross-spectral density estimate is obtained by first truncating the K-point third stage correlation function estimate $\hat{c}_{12}(m)$ to M points as

$$\hat{c}_{12}^{(g)}(m) = \begin{cases} \hat{c}_{12}(m); & -\frac{M}{2} \leqslant m < \frac{M}{2} \\ 0; & \text{otherwise.} \end{cases} \tag{25}$$

M and K are such that K/M is an integer. Now, the generalized spectrum estimate is obtained by taking the M-point DFT of $\hat{c}_{12}^{(g)}(m)$ and scaling it by a constant factor $1/\beta$. That is,

$$\hat{G}_{12}^{(g)}(k\Delta_K) = \frac{1}{\beta}\sum_{m=(-M/2)}^{(M/2)-1} \hat{c}_{12}^{(g)}(m)\, e^{-j(2\pi/M)km}. \tag{26}$$

In (26) β is a constant that depends on the window functions $w_1(n)$, $w_2(n)$, and $w_3(n)$, and also the parameters K, M, Q_1, Q_2, and R. We will choose β so as to conserve the signal power after the different operations. After deriving the expressions for the expected value of $\hat{G}_{12}^{(g)}(k\Delta_K)$, we will further discuss the choice of β. Before getting into the derivations, we will discuss how the five previously discussed methods fit into the generalized framework.

A. Special Cases

The generalized spectrum estimation algorithm will become the STUSE algorithm if we do not have any quadratic windowing in stage 3 and no truncation in stage 4. That is, $W_3(k\Delta_K)$ is a delta function and the final DFT length M is the same as the initial DFT length K. All the other algorithms, in one way or other, compute the average of the cross-PDS estimates obtained using the lth weighted segments of $\{x_1(n)\}$ and $\{x_2(n)\}$, and hence, for the rest of the section we will assume that $Q_1 = Q_2 = 0$.

To obtain the Blackman–Tukey spectrum estimates using the generalized method, we compute the first stage spectrum estimate using only one segment; i.e., use a rectangular linear window function of length equal to the number of data samples, pad at least the same number of zeros, and compute the auto/cross-PDS estimate of the signals involved. The inverse DFT of this estimate is an estimate of the auto/cross-correlation function of the signals. The BT estimate is now obtained by multiplying this correlation function estimate by an appropriately selected lag window function $w_3(m)$ which is defined to be zero except for $|M| \leqslant L$ where $2L+1$ is the number of lags for which the cross-correlation function is computed in (1). Note that while in the classical BT method the correlation function is directly estimated, in this theoretically equivalent approach, the correlation function estimates are obtained using DFT computations.

If we let $w_1(n) = w_2(n)$ be any arbitrary linear window function of length M (where zero padding has been taken care of by properly defining the window function), there is no truncation in stage 4 (i.e., $K = M$) and no quadratic modification in stage 3, the generalized algorithm becomes the WOSA method.

If in the above, $W_3(k\Delta_M)$ is a suitably selected quadratic window function, instead of a delta function, we get the lag-reshape method.

To see how Rader's algorithm fits into the generalized framework, choose $K = 4L_1$ in stage 1 and let $w_1(n)$ and $w_2(n)$ be rectangular window functions chosen as

$$w_1(n) = \begin{cases} 1; & n = 0, 1, \cdots, 3L_1 - 1 \\ 0; & \text{otherwise} \end{cases} \tag{27}$$

$$w_2(n) = \begin{cases} 1; & n = L_1, L_1+1, \cdots, 2L_1 - 1 \\ 0; & \text{otherwise.} \end{cases} \tag{28}$$

It is relatively straightforward to show that the estimate obtained using the above window functions after appropriate quadratic modification in stage 3, and truncation of the DFT length to $M = 2L_1$ in stage 4, is the same as Rader's estimate when the shift between adjacent segments $R = L_1$, provided one defines the initial and final segments of $\{x_1(n)\}$ and $\{x_2(n)\}$ appropriately.

IV. Statistical Considerations of the Generalized Spectrum Estimates

The derivation of the expected value of $\hat{G}_{12}^{(g)}(k\Delta_M)$ is straightforward, and the final result is given by

$$E\left\{\hat{G}_{12}^{(g)}(k\Delta_M)\right\} = G_{12}(k\Delta_M) \circledast W_e(k\Delta_M) \quad (29)$$

where $W_e(k\Delta_M)$ is the effective window function of the generalized estimation scheme, and it is the Fourier transform of the effective lag window function $r_e(m)$. The effective lag window function $r_e(m)$ is given by the following relationships:

$$r_e(m) = \begin{cases} \frac{1}{\beta} r_w(m); & \frac{-M}{2} \leq M < \frac{M}{2} \\ 0; & \text{otherwise,} \end{cases} \quad (30)$$

$$r_w(m) = \tilde{r}_{12}(m)\, w_3(m) \quad (31)$$

$$\tilde{r}_{12}(m) = \sum_{q=Q_1}^{Q_2} r_{12}(m + qR), \quad (32)$$

and $r_{12}(m)$ is the cross-correlation function of the linear window functions $w_1(n)$ and $w_2(n)$ obtained as

$$r_{12}(m) = \frac{1}{K} \sum_{n=0}^{K-1} w_1(n)\, w_2(n - m). \quad (33)$$

Note that $\tilde{r}_{12}(m)$, $r_w(m)$, and $r_e(m)$ are the effective lag window functions of the second, third, and fourth stages, respectively (i.e., the expected values of the cross-correlation function estimate at each stage is given by the product of the true correlation function and the effective lag window functions at each stage).

At this point, we are ready to discuss the choice of β in (26). β should be chosen such that the average power of the signals is preserved. For this we must choose β to be $r_e(0)$; i.e.,

$$\beta = r_e(0) = \frac{1}{M} \sum_{k=0}^{M-1} W_e(k\Delta_M). \quad (34)$$

We will now very briefly discuss the different approaches by which the generalized spectrum estimates can be made unbiased. At this point, it must be pointed out that none of the methods discussed in the paper will produce unbiased spectrum estimates if the cross-correlation function of the signals involved is zero for lags greater than half the length of the spectrum estimates [i.e., $c_{12}(m) = 0$ except for $-(M/2) \leq m < (M/2)$]. If this condition is satisfied, we can use any one of the following three approaches to obtain unbiased spectrum estimates.

1) We can choose $w_1(n)$ and $w_2(n)$ such that the cross-correlation function $r_{12}(m)$ of the two linear window functions is flat for $-(M/2) \leq m < (M/2)$ [i.e., $r_{12}(m) = r_{12}(0)$ for $-(M/2) \leq m < (M/2)$] and use no quadratic windowing at all after setting $Q_1 = Q_2 = 0$. Conceptually, Rader's algorithm yields unbiased spectrum estimates using this approach.

2) We may use identical linear window functions for both the channels [i.e., $w_1(n) = w_2(n)$], once again set $Q_1 = Q_2 = 0$ and choose $w_3(m)$ in such a way that the effective lag window function of the third stage spectrum estimate is flat for $-(M/2) \leq m < (M/2)$. Spectrum estimates using the lag-reshape and Blackman–Tukey algorithms can be made unbiased using this approach.

3) The third approach consists of choosing identical linear window functions for both channels and selecting the shift between adjacent segments R and the parameters Q_1 and Q_2 appropriately so that the spectrum estimates are unbiased. R should satisfy one of the following three conditions [21]:

a) $R = 1$,

b) the auto-PDS of $w_1(n)$ is zero for frequencies that are nonzero integer multiples of $1/R$, or

c) R is smaller than the "Nyquist" sampling rate of the "band-limited" window function $w_1(n)$.

Also Q_1 and Q_2 should be chosen such that

$$-Q_1 = Q_2 = \left\lfloor \frac{(L_1 + M)/2}{R} \right\rfloor \quad (35)$$

where $\lfloor(\cdot)\rfloor$ denotes the integer part of $(\cdot)$ and L_1 is the length of the window function $w_1(n)$ without taking into account the zeros padded to it. The STUSE algorithm can be made unbiased using this approach.

A. Variance Analysis

We will now derive expressions for the spectrum estimation variance of the generalized spectrum estimation algorithm. The derivations are based on signal spectra at discrete frequencies and, as a result, the expressions obtained can be exactly evaluated using numerical methods. Also, the derivations are made using the assumption that the sequences $\{x_1(n)\}$ and $\{x_2(n)\}$ belong to real, stationary, and zero-mean Gaussian time series.

For the purpose of analysis, we will assume that the K-point first stage spectrum estimate $\hat{G}_{12,q}(k\Delta_K)$ is obtained by first computing an L-point estimate $\hat{G}_{12,q}(k\Delta_L)$, and then sampling this estimate every L/K frequency bin. L is assumed to be an integer multiple of K and also at least twice as large as the number of data samples P. It is straightforward to show that the spectrum estimate obtained using the above process and using (14)–(16) are identical [20], [21]. The above assumption is useful in analysis, since all data samples may be correlated. (Note that we are not assuming that the spectral resolution of the estimation scheme is adequate to describe all the minute details of the signal spectrum.) This will become apparent during the derivations.

The derivation of the spectrum estimation variance is done on a stage-by-stage basis. Since the second, third, and fourth stages of the generalized spectrum estimation algorithm can be viewed as either additions of different estimates of the previous stage or convolution of the previous stage spectrum estimate with a known frequency domain window function, evaluation of the spectrum estimation variance at these stages can be easily done if one knows the covariance of the previous stage estimate for all combinations of frequencies. We will start the derivations by evaluating the covariance of the first stage

spectrum estimate at two arbitrary frequencies $k_1\Delta_L$ and $k_2\Delta_L$. The expected value of the product of $\hat{G}_{12,q_1}(k_1\Delta_L)$ and $\hat{G}^*_{12,q_2}(k_2\Delta_L)$ is obtained by expanding the product in terms of the signals and window functions involved and also the complex exponentials that are required during the Fourier transformation of the signals. This yields

$$\begin{aligned}
&E\left\{\hat{G}_{12,q_1}(k_1\Delta_L)\,\hat{G}^*_{12,q_2}(k_2\Delta_L)\right\} \\
&\quad=\frac{1}{N^2}\frac{1}{K^2}\sum_{l_1,l_2=0}^{N-1}\sum_{n_1,n_2,n_3,n_4=0}^{L-1} \\
&\quad\cdot E\Big\{\chi_1(l_1R+n_1)\,\chi_2\big((l_1+q_1)R+n_2\big) \\
&\qquad\chi_1(l_2R+n_3)\,\chi_2\big((l_2+q_2)R+n_4\big)\Big\} \\
&\quad\cdot w_1(n_1)\,w_2(n_2)\,w_1(n_3)\,w_2(n_4) \\
&\quad\cdot\exp\left\{-j\frac{2\pi}{L}\big(k_1(n_1-n_2)-k_2(n_3-n_4)\big)\right\}, \qquad (36)
\end{aligned}$$

where $w_1(n)$ and $w_2(n)$ are taken to be zero for $n > K$.

Because of Gaussianity, the fourth-order expectation in (36) can be broken up to sum of products of second-order expectations [23]. Expressing the resulting correlation functions as inverse DFT of the corresponding cross-spectral density {in these derivations, $G_{12}(\mu\Delta_L)$ is the DFT of the true cross-correlation function for lags in the range $\left[-\frac{L}{2},\frac{L}{2}\right)$; this may be different from the true cross-spectral density}, we can express the expectation on the right-hand side of (36) as

$$\begin{aligned}
&E\{\cdots\} \\
&=\frac{1}{L^2}\Bigg[\sum_{\mu,\nu=0}^{L-1}\Bigg\{G_{12}(\mu\Delta_L)\exp\left[j\frac{2\pi}{L}\mu(n_1-n_2-q_1R)\right] \\
&\cdot G^*_{12}(\nu\Delta_L)\exp\left[-j\frac{2\pi}{L}\nu(n_3-n_4-q_2R)\right]\Bigg\} \\
&+\sum_{\mu,\nu=0}^{L-1}\Bigg\{G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L) \\
&\cdot\exp\left[j\frac{2\pi}{L}\mu\big(n_1-n_2+(l_1-l_2)R\big)\right] \\
&\cdot\exp\left[-j\frac{2\pi}{L}\nu\big(n_2-n_4+(l_1+q_1-(l_2+q_2))R\big)\right]\Bigg\}\Bigg] \\
&+\sum_{\mu,\nu=0}^{L-1}\Bigg\{G_{12}(\mu\Delta_L)\,G^*_{12}(\nu\Delta_L) \\
&\cdot\exp\left[j\frac{2\pi}{L}\mu\big(n_1-n_4+(l_1-(l_2+q_2))R\big)\right] \\
&\cdot\exp\left[-j\frac{2\pi}{L}\nu\big(n_3-n_2+(l_2-(l_1+q_1))R\big)\right]\Bigg\}\Bigg]. \qquad (37)
\end{aligned}$$

Now, we can substitute (37) in (36) and expand (36) to obtain three terms, each one corresponding to one of the terms in (37). Of the three terms thus obtained, the first term is the product of the expected values of $\hat{G}_{12,q_1}(k_1\Delta_L)$ and $\hat{G}^*_{12,q_2}(k_2\Delta_L)$. The third term involves products of the form $W_1((k_1-\mu)\Delta_L)\,W_2(k_2+\mu)\Delta_L)$. All practical window functions have low-pass characteristics and, except at the folding frequencies [i.e., $k_1 = k_2 = 0$ or $(L/2)$], $W_1((k_1-\mu)\Delta_L)$ and $W_2((k_2+\mu)\Delta_L)$ will not overlap significantly on the μ scale [24]. As a result, we will neglect the third term throughout the variance analysis. At the folding frequencies, the second and third terms have comparable values and the overall variance will be approximately twice that obtained using our analysis. The above discussion implies that only the second term of (37) contributes to the covariance of $\hat{G}_{12,q_1}$ and $\hat{G}_{12,q_2}(k_2\Delta_L)$. Using this, and after some further simplifications, the covariance of $\hat{G}_{12,q_1}(k_1\Delta_K)$ and $\hat{G}_{12,q_2}(k_2\Delta_K)$ works out to

$$\begin{aligned}
&\mathrm{Cov}\left\{\hat{G}_{12,q_1}(k_1\Delta_K),\,\hat{G}_{12,q_2}(k_2\Delta_K)\right\}=\mathrm{Cov}\left\{\hat{G}_{12,q_1}(\kappa k_1\Delta_L),\,\hat{G}_{12,q_2}(\kappa k_2\Delta_L)\right\} \\
&\quad\approx\frac{1}{K^2}\Bigg\{\sum_{\mu,\nu=0}^{L-1}G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L) \\
&\quad\cdot\frac{W_1\big((\kappa k_1-\mu)\Delta_L\big)\,W_2^*\big((\kappa k_1-\nu)\Delta_L\big)\,W_1^*\big((\kappa k_2-\mu)\Delta_L\big)\,W_2\big((\kappa k_2-\nu)\Delta_L\big)}{L^2} \\
&\quad\cdot\xi^2(\mu-\nu)\exp\left[-j\frac{2\pi}{L}\nu(q_1-q_2)R\right]\Bigg\} \qquad (38)
\end{aligned}$$

where $\kappa = L/K$ and

$$\begin{aligned}
\xi^2(x)&=\frac{1}{N^2}\sum_{l_1,l_2,=0}^{N-1}\exp\left[j\frac{2\pi}{L}(l_1-l_2)Rx\right] \\
&=\left[\frac{\sin\frac{\pi}{L}NRx}{N\sin\frac{\pi}{L}Rx}\right]^2. \qquad (39)
\end{aligned}$$

Note that in deriving (38), we have made use of the fact that the K-point spectrum estimate is obtained by sampling the L-point estimate every $\kappa = L/K$ frequency bins. In the second term of (37), the exponential quantities $\exp[j(2\pi/L)\mu(l_1-l_2)R]$ and $\exp[-j(2\pi/L)\nu(l_1-l_2+q_1-q_2)R]$ take care of the time shift between the lth and l_2th segments and the (l_1+q_1)th and (l_2+q_2)th segments, respectively. (The same argument holds for the corresponding exponential quantities in the third term also.) If L is smaller than twice the data length $(2P)$, be-

cause of the periodic nature of the complex exponential functions, the above functions will not correctly reflect the dependence between segments which are more than $L/2$ (which is half the period of the complex exponentials) apart and will give wrong results—hence the need for assuming that $L \geq 2P$ for the analysis of the spectrum estimation variance.

Using the relationship between the first and second stage spectrum estimates as given by (18) and the covariance expression given by (38), we can obtain an expression for the covariance of the second stage spectrum estimates at frequencies $k_1\Delta_K$ and $k_2\Delta_K$ as

$$\begin{aligned}\text{Cov}\left\{\tilde{G}_{12}(k_1\Delta_K),\ \tilde{G}_{12}(k_2\Delta_K)\right\} &= \frac{1}{K^2}\sum_{\mu,\nu=0}^{L-1}\Big\{G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L)\,\xi^2(\mu-\nu) \\ &\cdot \zeta(\kappa k_1-\nu)\,\zeta^*(\kappa k_2-\nu) \\ &\cdot W_1\big((\kappa k_1-\mu)\Delta_L\big)\,W_2^*\big((\kappa k_1-\nu)\Delta_L\big) \\ &\cdot W_1^*\big((\kappa k_2-\mu)\,W_2(\kappa k_2-\nu)\big)\Big\},\end{aligned} \tag{40}$$

where

$$\zeta(x) = \left\{\frac{\sin\left(\frac{\pi}{L}(Q_2-Q_1+1)Rx\right)}{\sin\left(\frac{\pi}{L}Rx\right)}\right\} \cdot \exp\left[j\frac{\pi}{L}(Q_2+Q_1)Rx\right]. \tag{41}$$

The variance of the second stage spectrum estimate is given by substituting $k = k_1 = k_2$ in (40). That is,

$$\begin{aligned}\text{Var}\left\{\tilde{G}_{12}(k\Delta_K)\right\} &= \frac{1}{K^2}\sum_{\mu,\nu=0}^{L-1}\Big\{G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L) \\ &\cdot G_{w_1w_1}\big((\kappa k-\mu)\Delta_L\big)\,G_{w_2w_2}\big((\kappa k-\nu)\Delta_L\big) \\ &\cdot \xi^2(\mu-\nu)\left|\zeta(\kappa k-\nu)\right|^2,\end{aligned} \tag{42}$$

where $G_{w_1w_1}(k\Delta_L)$ and $G_{w_2w_2}(k\Delta_L)$ are the L-point auto-PDS of the window functions $w_1(n)$ and $w_2(n)$, respectively.

The third stage spectrum estimate $\hat{G}_{12}(k\Delta_K)$ is a frequency smoothed version of the second stage spectrum estimate given by (19) and (20). Combining (20) and (40), the covariance of the third stage spectrum estimates $\hat{G}_{12}(k_1\Delta_K)$ and $\hat{G}_{12}(k_2\Delta_K)$ works out to

$$\begin{aligned}\text{Cov}\left\{\hat{G}_{12}(k_1\Delta_K),\ \hat{G}_{12}(k_2\Delta_K)\right\} &= \frac{1}{K^2}\sum_{\sigma_1,\sigma_2=0}^{K-1}\text{Cov}\left\{\tilde{G}_{12}\big((k_1-\sigma_1)\Delta_K\big),\right. \\ &\left.\tilde{G}_{12}\big((k_2-\sigma_2)\Delta_K\big)\right\} W_3(\sigma_1\Delta_K)\,W_3^*(\sigma_2\Delta_K)\end{aligned} \tag{43}$$

$$= \frac{1}{K^2}\sum_{\mu,\nu=0}^{L-1} G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L)\,\xi^2(\mu-\nu)\,\Gamma(\kappa k_1-\mu,\ \kappa k_1-\nu)\,\Gamma^*(\kappa k_2-\mu,\ \kappa k_2-\nu), \tag{44}$$

where

$$\Gamma(x, y) = \frac{1}{LK}\sum_{\sigma=0}^{K-1} W_1\big((x-\kappa\sigma)\Delta_L\big)\cdot W_2^*\big((y-\kappa\sigma)\Delta_L\big)\,W_3(\sigma\Delta_K)\,\zeta(y-\kappa\sigma). \tag{45}$$

Once again, the estimation variance for $\hat{G}_{12}(k\Delta_M)$ can be obtained from (44) by substituting $k_1 = k_2 = k$ in this equation.

At this point, we are in a position to obtain the estimation variance of the generalized spectrum estimate. It is easy to see that $\hat{G}_{12}^{(g)}(k\Delta_M)$ can be obtained by sampling $\hat{G}_{12}(k\Delta_K)$ every $\lambda = (K/M)$ frequency bins after convolving it with a quadratic window function $W_R(k\Delta_K)$ that is obtained as the Fourier transform of the rectangular window function $w_R(m)$ given by

$$w_R(m) = \begin{cases}1; & -\dfrac{M}{2} \leq m < \dfrac{M}{2} \\ 0; & \text{otherwise}\end{cases} \tag{46}$$

and then scaling the frequency smoothed version by a factor $1/\beta$. That is,

$$\hat{G}_{12}^{(g)}(k\Delta_M) = \frac{1}{\beta}\hat{G}_{12}(\lambda k\Delta_K) \circledast W_R(\lambda k\Delta_K); \qquad k = 0, 1, \cdots, M-1. \tag{47}$$

Then,

$$\begin{aligned}\text{Var}\left\{\hat{G}_{12}^{(g)}(k\Delta_K)\right\} &= \frac{1}{\beta^2}\frac{1}{K^2}\sum_{\rho_1,\rho_2=0}^{K-1}\text{Cov}\left\{\hat{G}_{12}\big((\lambda k-\rho_1)\Delta_K\big),\right. \\ &\left.\hat{G}_{12}\big((\lambda k-\rho_2)\Delta_K\big)\right\} \cdot W_R(\rho_1\Delta_K)\,W_R^*(\rho_2\Delta_K)\end{aligned} \tag{48}$$

$$= \frac{1}{\beta^2K^2}\sum_{\mu,\nu=0}^{L-1} G_{11}(\mu\Delta_L)\,G_{22}(\nu\Delta_L)\left|\alpha(\kappa\lambda k-\mu,\ \kappa\lambda k-\nu)\right|^2\xi^2(\mu-\nu), \tag{49}$$

where $\alpha(x, y)$ is given by

$$\alpha(x, y) = \frac{1}{K}\sum_{\rho=0}^{K-1}\Gamma(x-\kappa\rho,\ y-\kappa\rho)\,W_R(\rho\Delta_K). \tag{50}$$

The main advantage of (49) is that it can be evaluated exactly using numerical methods because there are only arithmetical operations here. However, for large values

of L, it may still be very time consuming to evaluate this quantity. Simplifications of (49) are possible if we assume that the spectra of the signals involved do not vary much within the range of the statistical bandwidth of the effective window function [3]. Under this assumption, the normalized variance defined as

$$\mathrm{NVAR}\left\{\hat{G}_{12}^{(g)}(k\Delta_M)\right\} = \frac{\mathrm{Var}\left\{\hat{G}_{12}^{(g)}(k\Delta_M)\right\}}{E\left\{\hat{G}_{11}^{(g)}(k\Delta_M)\right\} E\left\{\hat{G}_{22}^{(g)}(k\Delta_M)\right\}} \tag{51}$$

gets approximated to

$$\mathrm{NVAR}\left\{\hat{G}_{12}^{(g)}(k\Delta_M)\right\} = \frac{1}{\beta^2}\frac{1}{K}\frac{1}{N}\sum_{l=1-N}^{N-1}\left(1-\frac{|l|}{N}\right)\sum_{q_1,q_2=Q_1}^{Q_2} \cdot \sum_{t=-(K/2)}^{(K/2)-1}\psi(t, l, q_1-q_2)\, w_3^2(t-q_1R) \cdot w_R(t-q_1R), \tag{52}$$

where

$$\psi(t, l, q) = \frac{1}{K}\sum_{t_1=0}^{K-1} w_1(t_1+t)\, w_1(t_1+t-lR) \cdot w_2(t_1)\, w_2(t_1+(l+q)R). \tag{53}$$

For details of the derivation, the reader is referred to [20]. The derivations in this section follow those in [21] and [27] very closely.

We now present an example that verifies the variance expressions derived above. The experiment presented here was done on synthetic data obtained by passing zero mean, white, Gaussian pseudorandom signals through a second-order filter whose transfer function is given by

$$H(z) = \frac{0.367}{1 - z^{-1} + 0.4z^{-2}}. \tag{54}$$

The results are averages of 2240 independent spectrum estimates based on 512 data samples each. The parameters of the spectrum estimation algorithm employed were as follows. The linear window functions $w_1(n)$ and $w_2(n)$ were both identical rectangular window functions of length 64 samples each, there was no overlap (i.e., $R = 64$), $-Q_1 = Q_2 = 1$, and the FFT length was 256. Also, there was no truncation in the final stage (i.e., $K = M = 256$). Finally, the quadratic window function used was such that the effective window function was the same as that for the STUSE algorithm using 64-point Hanning windows, 256-point FFT's, $-Q_1, = Q_2 = 2$, and $R = 16$.

Fig. 1(a) displays the true autopower spectral density of the signals along with the ensemble averages of the estimates. Fig. 1(b) and (c) shows plots of the theoretical variance along with the ensemble variance and the theoretical normalized variance along with the ensemble normalized variance, respectively. The theoretical plots in Fig. 1(b) and (c) were computed using the approximate expression in (52). From this figure, we can make the following observations. At the folding frequencies, the ensemble variance is approximately twice that predicted by (52). This result is in accordance with the discussion after (37). The theoretical and ensemble variances in Fig. 1(b) and (c) show close match. On an average, the ensemble normalized variance is approximately two percent smaller than that predicted by (52). The small discrepancies between the theoretical and ensemble variances can be attributed to a combination of the approximations made during the derivations, the statistical variability of the experiment, and imperfections in the pseudorandom number generator.

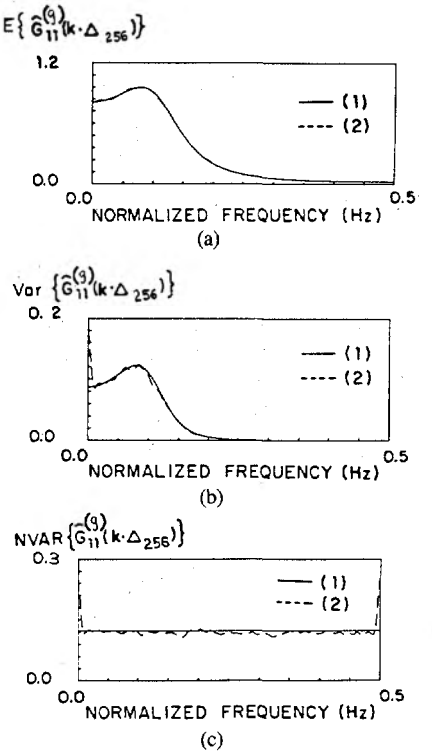

Fig. 1. Example used to verify the derivations for estimation variance of the generalized nonparametric spectrum estimation algorithm. (a) (1) True spectrum, (2) ensemble mean of estimates. (b) (1) Theoretical and (2) ensemble variances. (c) (1) Theoretical and (2) ensemble normalized variances.

Finally in this section, we will make a comparative study of the estimation stability of different unbiased methods, when all of them have the same effective window function. The methods compared are the Blackman–Tukey algorithm, the lag-reshape method with several different initial window lengths, the STUSE algorithm, and Rader's approach. In each situation, the effective window was the same as that for the STUSE algorithm that uses 32-point Hanning window functions, 50 percent overlaps, and $-Q_1 = Q_2 = 3$. The data length is assumed to be 1024 samples. Note that the estimation variance is independent of the final FFT length. In Table I we have tabulated the theoretical normalized variance obtained using

TABLE I
APPROXIMATE NORMALIZED VARIANCES FOR DIFFERENT UNBIASED METHODS WHEN 1024 DATA SAMPLES ARE USED AND THE EFFECTIVE WINDOWS ARE IDENTICAL

Method	Parameters	Approximate Normalized Variance	Remarks
Blackman-Tukey	—	0.105	—
Lag-reshape	$K = 128$	0.132	Segment length = $K/2$
	$K = 256$	0.114	Rectangular linear window
	$K = 512$	0.108	No segment overlap
	$K = 2048$	0.105	$K = 2048$ is same as BT
STUSE	$-Q_1 = Q_2 = 3, R = 16$	0.104	32-point Hanning window
Rader	—	0.102	64-point initial segment

(52) for the different methods along with the pertinent parameters. We can see that all the methods, except the lag-reshape approach with smaller segment sizes ($K = 128$ and 256), exhibit more or less similar estimation variances.

The poorer performance of the lag-reshape algorithm for smaller segment lengths can be intuitively explained in the following manner. Note that for the LR method (and also for the BT approach), unbiased estimates are obtained using lag (quadratic) window functions. This lag window function is obtained as the quotient of the effective lag window function and the autocorrelation function of the linear rectangular window function that is used. Since for small rectangular window functions, the autocorrelation function decreases to zero very rapidly, the corresponding lag window function increases very rapidly, which in turn increases the variance of the correlation function estimates at these lags. This increased variance reduces the stability of the corresponding spectrum estimates. Since both STUSE and Rader's algorithms do not make use of lag windows that increase with lag, their estimation variances are not disproportionately affected. Also, when the segment sizes are much larger than the size of the effective lag window function, the rate at which the lag window function increases with increasing lags is small, and thus, there is no undue increase in estimation variance.

V. SPECTRAL RESOLUTION AND LEAKAGE SUPPRESSION

In this section, we will discuss some subtle aspects of spectral resolution and spectral leakage suppression capability of the nonparametric spectrum estimation techniques. The spectral resolution of all the methods discussed here is limited by the inverse of the duration of the time series. However, there are at least two different approaches by which one can improve the spectral resolution to beyond what is obtained using the linear window functions.

We can use a quadratic window function in such a way that the effective window function has smaller statistical bandwidth than without quadratic windowing. However, this approach is not recommended since the estimation variance is increased disproportionately to the improvement in resolution. This is especially true if the lag window function needs to be very large for higher lags in order to achieve the required resolution. Also, the improvement in spectral resolution is limited to the inverse of the duration of the cross-correlation function of the linear window functions applied to the two channels.

The second approach to increasing the spectral resolution is the method employed by the STUSE algorithm, where the cross-PDS estimates are based on not only the averages of the sample cross-PDS of the lth windowed segments of $\{x_1(n)\}$ and $\{x_2(n)\}$, but also on the averages of the sample cross-PDS of the lth segment of $\{x_1(n)\}$ and the $(l + q)$th segment of $\{x_2(n)\}$ for different values of q ranging from Q_1 to Q_2 [see (6)–(8)]. This approach has two obvious advantages.

1) This method produces spectrum estimates with better stability than the first approach.

2) One can choose arbitrarily small linear window function with specific sidelobe structure and vary the parameters Q_1 and Q_2 to obtain good spectral resolution without sacrificing the spectral leakage suppression capability [21]. Note that this effectively delinks the spectral resolution and leakage suppression aspects. We will discuss this fact further in a short while.

In the rest of the section, we will discuss the spectral leakage suppression capability of unbiased nonparametric spectrum estimation algorithms. Spectral leakage is caused by strong sidelobes of the window functions. It is well known that suitably selected linear window functions will suppress spectral leakage effectively [6], [15], [19]. It is also well accepted that for the spectrum estimates to exhibit good spectral leakage suppression, the effective window function should have good sidelobe structure. It was shown in [27] and [28] that by choosing the quadratic window function properly, one can compensate for the bad sidelobe behavior of the linear window function that is used. However, there has been considerable controversy about whether excellent sidelobe structures achieved using quadratic functions can effectively reduce spectral leakage [4], [32], [37].

For periodograms, it was pointed out in [32] that the leakage suppression brought about by the improved side-

lobe structure is masked by estimation noise. Similarly, for segment averaging-type spectrum estimation techniques, except for the case where the estimation noise levels are very small (i.e., when there is a large number of segments over which the spectrum estimates are averaged), this estimation noise will mask any reduction in the spectral leakage that can possibly be obtained using a quadratic window function [22].

The arguments in [22] can be extended to the case of unbiased spectrum estimation algorithms also. When the signals have a large dynamic range (when large peaks and valleys are bunched together at closely spaced frequencies) and when rectangular linear window functions (or other linear window functions with bad sidelobe structure) are used, the bad sidelobe structure of the linear window function will mask the weaker details of the spectrum. Using (38), it is easy to show that the variance of the first stage spectrum estimate obtained using (16) (i.e., using only one segment for each channel) is proportional to the product of the expected values of the individual autopower density spectrum estimate. Segment averaging, while reducing the noise level, will, in general, not bring the noise levels down to that of the weaker spectral components. As a result, at frequencies with weak spectral power, since the expected value of the spectrum estimate may be several magnitudes larger than the spectrum itself, the estimation noise will also be several magnitudes larger than the signal spectrum. When an appropriate quadratic window function is applied to the above spectrum estimate, the magnitude of the expected value of the spectrum estimate is reduced to the correct levels (as a result of the sidelobe compensation brought about by the quadratic window function). Unfortunately, the estimation noise is not proportionately reduced [22], [32]. This results in the masking of the weak details by the spectrum estimation noise. The only way this masking will not occur is if the estimation noise is small at frequencies where the signal strength is weak. This can happen in two ways: 1) when the number of data samples are much larger than the required spectral resolution, and 2) when the expected value of the spectrum estimate has the same order of magnitude as the weak signal level. (Note that this can be achieved by using a linear window function with good sidelobe structure.)

We must also point out that the above discussion applies not to the average behavior of the spectrum estimation scheme but to each individual estimate. On an average, all the methods will perform in the same way if the effective window functions are the same. However, because of the effect of estimation errors, their performances will differ considerably for individual estimates.

Among the different unbiased methods discussed in this paper, only the STUSE algorithm uses linear window functions (different from rectangular window functions). Also, it has been shown in [21] that the effective window function of the STUSE algorithm has better sidelobe structure than that of the linear window function used. As a result, among all the unbiased nonparametric spectrum estimation methods discussed here, only the STUSE algorithm exhibits effective spectral leakage suppression when the data lengths involved are small.

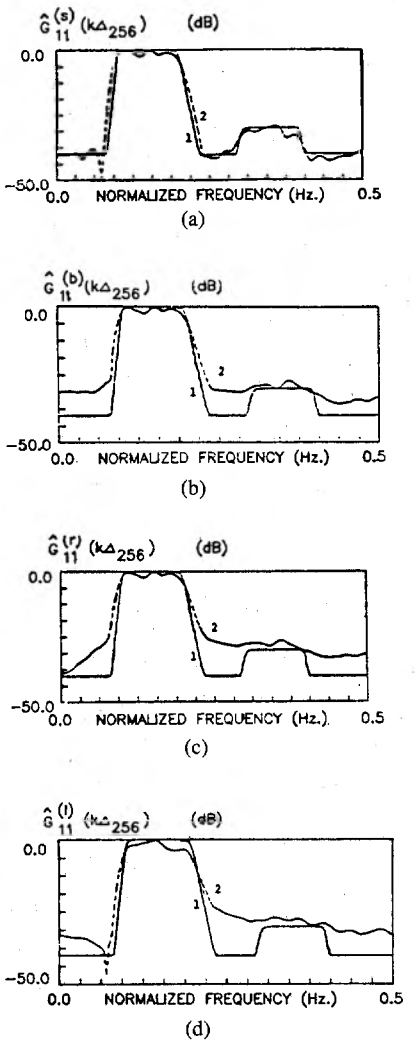

Fig. 2. Comparison of spectral leakage suppression capabilities of several unbiased nonparametric spectrum estimation algorithms. (a) STUSE, (b) Blackman–Tukey, (c) Rader, and (d) lag-reshape. (1) is the true spectrum, (2) is the estimated spectrum.

We will present a simulation example to demonstrate the above result. The true spectrum of the signal used in this example is plotted as curve 1 in Fig. 2(a)–(d). The signal was obtained using three independent zero-mean, white, Gaussian sequences of variance 1, 0.001, and 0.0001, respectively, sampled at 2000 Hz. The first two sequences were processed through tenth-order Butterworth bandpass filters with passbands 200–400 Hz and 600–800 Hz, respectively, and were added together along with the third sequence.

In this example, we will compare the 256-point auto-PDS estimates computed from 512 samples of this signal, obtained using the STUSE algorithm that uses 64-point Hanning windows, $R = 16$, and $-Q_1 = Q_2 = 1$; Blackman–Tukey approach; Rader's algorithm with 128-point data segments; and the lag-reshape algorithm that employs 128-point data segments. Wherever applicable, the

quadratic window functions were such that the effective window functions were the same as that of the STUSE algorithm using the same parameters as in this example. The auto-PDS estimates in decibels, computed using 512 data samples for each of these methods, are plotted as curve 2 in Fig. 2(a)–(d). We can see that only the STUSE algorithm picks up the smaller of the two peaks, demonstrating that only this method exhibits effective spectral leakage suppression, in spite of the fact that all the approaches had the same effective window function. For this example, the estimation variance for each method was evaluated at 500 Hz using (49) and a piecewise linear approximation for the true spectrum. The results showed that the estimation variances for the lag-reshape, Rader's, and Blackman–Tukey methods were larger than that of the STUSE algorithm by 18.2, 16.3, and 9.8 dB, respectively. Note that the better the sidelobe structure is before the quadratic window is applied, the smaller the final estimation variance is. As discussed earlier, if the time series involved is of a much larger duration, all the methods will exhibit good spectral leakage suppression.

VI. Concluding Remarks

The purpose of this paper was many-fold. First of all, we wanted to present a unified overview of nonparametric spectrum estimation techniques. Toward this goal, a generalized nonparametric spectrum estimation algorithm was introduced. This method contained as special cases the Blackman–Tukey, WOSA, lag-reshape, STUSE, and Rader's algorithms. Also, the generalization presented in the paper unifies more methods than that proposed by Nuttall and Carter [27], [28]. Expressions for the spectrum estimation variance of the generalized algorithm were derived under Gaussian assumptions. We also presented a simulation example to verify the theoretical derivations. All the derivations in the paper were done in terms of the signal spectra at discrete frequencies and, as a result, the expressions can be evaluated exactly using numerical methods. It must be pointed out that the estimation variance can be obtained by appropriately sampling the expressions derived using continuous frequency variables. However, the method developed here shows that numerical evaluation of these expressions may require knowledge of signal spectra at discrete frequencies that are much closer than those that are evaluated during spectrum estimation.

Unbiased nonparametric spectrum estimation algorithms were discussed in some detail in this paper. In Section IV we discussed three different approaches to obtaining unbiased spectrum estimates and pointed out examples from literature that work in each of these methods. The stability aspects of different nonparametric methods were compared in the same section, and it was shown that within the accuracy of the approximate variance formula, all four methods are equally stable, except for the case when unbiased spectrum estimates were obtained using the lag-reshape approach with very small initial segment sizes.

We discussed some subtle aspects of spectral leakage suppression in Section V. It was shown that good sidelobe structure by itself is not enough to provide effective spectral leakage suppression. We also need basic linear windowing of the time series with window functions that have good sidelobe structure. Otherwise, even when the effective window functions have good sidelobe structure, the correlated estimation noise will mask the spectral leakage suppression that is attainable. Because of this, among the many unbiased spectrum estimation schemes that are popularly used, only the STUSE algorithm exhibits effective spectral leakage suppression capability when the time series is of short duration.

Finally, we believe it is appropriate to discuss the suitability of different algorithms in different situations. First of all, if the dynamic range of the spectrum being estimated is small, we must realize that there is little possibility of problems arising due to spectral leakage. Thus, we can avoid linear windowing altogether (i.e., use only rectangular window functions) and save a great deal on the number of multiplications involved. In such situations, we can choose the Blackman–Tukey algorithm or one of the segment averaging techniques. If a segment averaging method is chosen, we also need no segment overlap. Also, if unbiased spectrum estimates are required, Rader's algorithm is the best, since this method is computationally the least expensive.

However, when the signal spectra have large peaks and valleys close together, we must make sure that the algorithm exhibits good spectral leakage suppression. If the data length is large, as discussed in Section V, we can use suitably selected quadratic window functions and obtain good sidelobe structure, which would in turn provide effective spectral leakage suppression. The suggested algorithm for such a situation is the lag-reshape approach. If unbiased estimates are needed, one may use Rader's algorithm with suitable quadratic windowing, or one may employ the lag-reshape method with large data segments. Finally, if the dynamic range is large and the data length is relatively short, one must use the WOSA method with a linear window function that has good sidelobe structure. If unbiased estimates are also required under these conditions, STUSE algorithm is the suggested approach.

We must point out here that, throughout this paper, we have assumed that the data length is large enough to give the desired spectral resolution. If this is not the case, one must resort to one or the other of the different parametric spectrum estimation techniques that are available, or some of the more complex nonparametric approaches as in [33].

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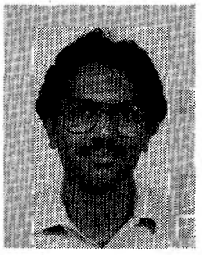

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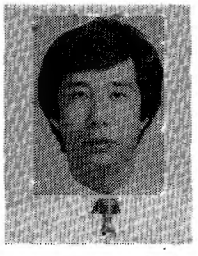

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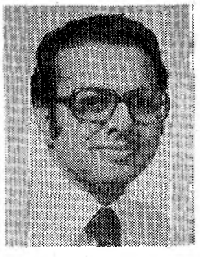

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